Quantum Mechanics I

Week 12 (Solutions)

Spring Semester 2025

1 The Action of the Angular Momentum Operator in Position Representation

Find the action of the z-component of the angular momentum on a ket state $|\alpha\rangle$ in the position representation, i.e.

$$\langle \mathbf{x}'|L_z|\alpha\rangle = -i\hbar\frac{\partial}{\partial\phi}\langle \mathbf{x}'|\alpha\rangle. \tag{1.1}$$

Hint: Use the standard spherical-coordinate transformation

$$x' = r\cos\phi\sin\theta, \qquad y' = r\sin\phi\sin\theta, \qquad z' = r\cos\theta.$$

We are interested in computing the matrix element $\langle \mathbf{x}'|L_z|\alpha\rangle$. The angular momentum is defined as $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$ and its components are given by $L_i = \epsilon_{ijk}r_j p_k$. The z-component is given by $L_z = xp_y - yp_z$. Thus, we have:

$$\begin{aligned}\langle \mathbf{x}'|\hat{L}_z|\alpha\rangle &= \langle \mathbf{x}'|(xp_y - yp_z)|\alpha\rangle \\ &= x'\big[-i\hbar\frac{\partial}{\partial y'}\langle \mathbf{x}'|\alpha\rangle\big] - y'\big[-i\hbar\frac{\partial}{\partial x'}\langle \mathbf{x}'|\alpha\rangle\big].\end{aligned}$$

To proceed, we first express the derivatives in spherical coordinates using the chain rule:

$$\frac{\partial}{\partial x'} = \frac{\partial r}{\partial x'}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x'}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x'}\frac{\partial}{\partial \phi}, \qquad \frac{\partial}{\partial y'} = \frac{\partial r}{\partial y'}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y'}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y'}\frac{\partial}{\partial \phi}. \tag{1.2}$$

Using the hint provided in the instruction, we can determine the derivatives of the spherical coordinates with respect to the cartesian ones. We thus find:

$$\begin{aligned}\langle \mathbf{x}'|\hat{L}_z|\alpha\rangle &= -i\hbar\Big[r\cos\phi\sin\theta\Big(\sin\phi\sin\theta\,\frac{\partial}{\partial r}\Big) - r\sin\phi\sin\theta\Big(\cos\phi\sin\theta\,\frac{\partial}{\partial r}\Big)\Big]\langle \mathbf{x}'|\alpha\rangle \\ &\quad - i\hbar\Big[r\cos\phi\sin\theta\Big(-\frac{\sin\phi\cos\theta}{r}\,\frac{\partial}{\partial\theta}\Big) - r\sin\phi\sin\theta\Big(-\frac{\cos\phi\cos\theta}{r}\,\frac{\partial}{\partial\theta}\Big)\Big]\langle \mathbf{x}'|\alpha\rangle \\ &\quad - i\hbar\Big[r\cos\phi\sin\theta\Big(\frac{\cos\phi\,\csc\theta}{r}\,\frac{\partial}{\partial\phi}\Big) - r\sin\phi\sin\theta\Big(-\frac{\sin\phi\,\csc\theta}{r}\,\frac{\partial}{\partial\phi}\Big)\Big]\langle \mathbf{x}'|\alpha\rangle \\ &= -i\hbar\,\sin\theta\,\csc\theta\,\frac{\partial}{\partial\phi}\langle \mathbf{x}'|\alpha\rangle \\ &= -i\hbar\,\frac{\partial}{\partial\phi}\langle \mathbf{x}'|\alpha\rangle,\end{aligned}$$

and thus yield the final result.

2 Rotations about the z-axis

Let $\hat{\mathbf{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$ be the angular-momentum operator of a particle. In the position representation, written in spherical coordinates (r, θ, ϕ), we have

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi} \,. \tag{2.1}$$

Let

$$\hat{D}_z(\alpha) = e^{-i\alpha \hat{L}_z/\hbar} \tag{2.2}$$

be the rotation operator for an angle α about the z-axis, acting in the Hilbert space of the particle.

(a) Let $\psi'(r, \theta, \phi) = \hat{D}_z(\alpha)\psi(r, \theta, \phi)$ be the wave-function obtained by applying $\hat{D}_z(\alpha)$ to an arbitrary state $\psi(r, \theta, \phi)$. Show that

$$\psi'(r, \theta, \phi) = \psi\big(r, \theta, \phi - \alpha\big). \tag{2.3}$$

where the prime corresponds to the rotated state.

We expand the exponential of the rotation operator in its Taylor series,

$$\begin{aligned}
\psi'(r, \theta, \phi) &= D_z(\alpha)\psi(r, \theta, \phi) \\
&= e^{-i\alpha\, \hat{L}_z/\hbar}\psi(r, \theta, \phi) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i\alpha}{\hbar}\right)^n \hat{L}_z^n \psi(r, \theta, \phi) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(-\alpha)^n \partial_\phi^n \psi(r, \theta, \phi) \\
&= \psi\big(r, \theta, \phi - \alpha\big),
\end{aligned}$$

where in the last step we used the Taylor expansion of $\psi(r, \theta, \phi - \alpha)$ with respect to $\alpha = 0$.

(b) Express $\psi'(x, y, z)$ as a function of $\psi(x, y, z)$ and α.

We explicitly write the dependence of (r, θ, ϕ) as a function of (x, y, z):

$$\begin{aligned}
&\tilde{\psi}\big(x(r, \theta, \phi),\, y(r, \theta, \phi),\, z(r, \theta, \phi)\big) = \psi\big(x(r, \theta, \phi - \alpha),\, y(r, \theta, \phi - \alpha),\, z(r, \theta, \phi - \alpha)\big) \\
&\quad = \psi\big(r \sin\theta \cos(\phi - \alpha),\, r \sin\theta \sin(\phi - \alpha),\, r\cos\theta\big) \\
&\quad = \psi\big(r \sin\theta \cos\phi \cos\alpha + r \sin\theta \sin\phi \sin\alpha,\, r \sin\theta \sin\phi \cos\alpha - r \sin\theta \cos\phi \sin\alpha,\, r \cos\theta\big) \\
&\quad = \psi\big(x \cos\alpha + y \sin\alpha,\, y \cos\alpha - x \sin\alpha,\, z\big).
\end{aligned}$$

We have used the trigonometric identity $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$ and $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$. We remark that the result corresponds to the function ψ at coordinates rotated by an angle $-\alpha$ around the z-axis.

(c) Consider the three states described by the wave functions

$$\psi_x(\mathbf{r}) = x\, f(r),\ \psi_y(\mathbf{r}) = y\, f(r),\ \psi_z(\mathbf{r}) = z\, f(r), \tag{2.4}$$

where $f(r)$ vanishes for $r \to +\infty$ so that the states are normalisable. These three states are mutually orthogonal and thus form a basis of a subspace $\mathcal{H}_1$ of dimension 3 of the Hilbert space $\mathcal{H}$. Show that each $\psi_j(\mathbf{r})$ is an eigenstate of $\hat{L}_j$ $(j = x, y, z)$ and calculate the corresponding eigenvalues. *Hint: You may use* $[\hat{L}_j, \hat{r}_k] = i\hbar\, \epsilon_{jkl}\hat{r}_l$.

We will use the commutation relation for angular momentum and position as provided in the instruction. For the x compoments of angular momentum and position, the commutator is zero, i.e. $[\hat{L}_x, \hat{x}] = 0$. We can use this, together with the property of the position operator in the position representation, in the following way:

$$\begin{aligned}
\hat{L}_x\psi_l(r) &= L_x x\, f(r) \\
&= \hat{L}_x\hat{x}\, f(r) \\
&= \hat{x}\hat{L}_x f(r) \\
&= x\hat{L}_x f(r) \\
&= x(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y) f(r) \\
&= -i\hbar x(y\partial_z - z\partial_y) f(r) \\
&= -i\hbar x\left(y\frac{\partial r}{\partial z} - z\frac{\partial r}{\partial y}\right)\frac{\partial f(r)}{\partial r} \\
&= -i\hbar x\left(\frac{yz}{r} - \frac{zy}{r}\right)\frac{\partial f(r)}{\partial r} = 0.
\end{aligned}$$

and we have used the fact that $r = \sqrt{x^2 + y^2 + z^2}$. The function $\psi_x(\mathbf{r})$ is therefore an eigenfunction of the operator L^x with eigenvalue zero. Analogously, we calculate $\hat{L}_y\psi_y(\mathbf{r})$ and $\hat{L}_z\psi_z(\mathbf{r})$ and find again zero-eigenvalues, respectively.

(d) Using the result of part (b), show that $\mathcal{H}_1$ is invariant under the action of $\hat{D}_z(\alpha)$. Express the matrix associated with $\hat{D}_z(\alpha)$ in the basis $\{\psi_x, \psi_y, \psi_z\}$.

We can use the general result obtained from part (a). For $\psi_x(\mathbf{r})$, we obtain:

$$\begin{aligned}
\hat{D}_z(\alpha)\psi_x(\mathbf{r}) &= \hat{D}_z(\alpha)\, x\, f(r) \\
&= (x\cos\alpha + y\sin\alpha)\, f(r) \\
&= \psi_x(r)\cos\alpha + \psi_y(r)\sin\alpha.
\end{aligned}$$

Likewise, for $\psi_y(\mathbf{r})$

$$\begin{aligned}
\hat{D}_z(\alpha)\psi_y(\mathbf{r}) &= \hat{D}_z(\alpha)\, y\, f(r) \\
&= (y\cos\alpha - x\sin\alpha)\, f(r) \\
&= \psi_y(r)\cos\alpha - \psi_x(r)\sin\alpha,
\end{aligned}$$

and $\psi_z(\mathbf{r})$:

$$\hat{D}_z(\alpha)\psi_z(\mathbf{r}) = \hat{D}_z(\alpha)\, z\, f(r) = \psi_z(r). \tag{2.5}$$

We see that the application of $\hat{D}_z(\alpha)$ to the three functions always gives linear combinations of the same three functions, which shows that the subspace $\mathcal{H}_1$ is indeed invariant under the action of $\hat{D}_z(\alpha)$. The expressions we have just obtained also define the matrix form of $\hat{D}_z(\alpha)$:

$$\hat{D}_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.6}$$

3 Angular Momenta and Uncertainty

The commutator relations for the angular momenta

$$[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k \tag{3.1}$$

imply an important set of uncertainty relations among the angular momenta, namely:

$$\Delta L_x \, \Delta L_y \;\geq\; \frac{\hbar}{2}\,|\langle \hat{L}_z\rangle|, \qquad \Delta L_y \, \Delta L_z \;\geq\; \frac{\hbar}{2}\,|\langle \hat{L}_x\rangle|, \qquad \Delta L_z \, \Delta L_x \;\geq\; \frac{\hbar}{2}\,|\langle \hat{L}_y\rangle|. \tag{3.2}$$

Now consider a particle in a normalized eigenstate of $\hat{L}^2$ and $\hat{L}_z$.

(a) Show that in this case the expectation values of the x and y components of the angular momentum are zero, i.e. $\langle L_x\rangle = \langle L_y\rangle = 0$.

We use the ladder operators $L_\pm$ to show this. The x and y components can be expressed in terms of ladder operators:

$$L_x = \frac{1}{2}\Big[L_+ + L_-\Big], \quad L_y = \frac{1}{2i}\Big[L_+ - L_-\Big]. \tag{3.3}$$

First, for the x component, we have:

$$\begin{aligned}\langle L_x\rangle &= \frac{1}{2}\left[\langle l,m|\, L_+\, |l,m\rangle + \langle l,m|\, L_-\, |l,m\rangle\right] \\ &\sim \langle l,m|l,m+1\rangle + \langle l,m|l,m-1\rangle \\ &= 0.\end{aligned}$$

due to the orthogonality condition of the eigenstates $|l,m\rangle$. Similarly, for the y component, we have:

$$\begin{aligned}\langle L_y\rangle &= \frac{1}{2i}\left[\langle l,m|\, L_+\, |l,m\rangle - \langle l,m|\, L_-\, |l,m\rangle\right] \\ &\sim \langle l,m|l,m+1\rangle - \langle l,m|l,m-1\rangle \\ &= 0,\end{aligned}$$

and we used again the orthogonality condition of $|l,m\rangle$.

(b) Show that

$$\langle L_y^2\rangle = \langle L_x^2\rangle = \frac{\hbar^2}{2}\left[l(l+1) - m^2\right]. \tag{3.4}$$

We express the operator $\hat{L}_x^2$ as

$$\hat{L}_x^2 = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)(\hat{L}_+ + \hat{L}_-) = \frac{1}{4}(\hat{L}_+^2 + \hat{L}_-\hat{L}_+ + \hat{L}_+\hat{L}_- + \hat{L}_-^2). \tag{3.5}$$

The first and last terms are of no consequence because when we "sandwich" them between $|l, m\rangle$ to find the expectation value we get zero due to the orthonormality of $|l, m\rangle$. Thus, we are left with

$$\langle l, m|L_x^2|l, m\rangle = \frac{1}{4}\Big[\langle l, m|\, L_-L_+\, |l, m\rangle + \langle l, m|\, L_+L_-\, |l, m\rangle\big]. \tag{3.6}$$

We can deal with what is left by doing a little commutator algebra:

$$\begin{aligned}
\hat{L}_\pm\hat{L}_\mp &= (\hat{L}_x \pm i\hat{L}_y)(\hat{L}_x \mp i\hat{L}_y) \\
&= \hat{L}_x^2 + \hat{L}_y^2 \pm i\hat{L}_y\hat{L}_x \mp i\hat{L}_x\hat{L}_y \\
&= \hat{L}^2 - \hat{L}_z^2 \pm i[\hat{L}_x, \hat{L}_y] \\
&= \hat{L}^2 - \hat{L}_z^2 \pm \hbar\hat{L}_z,
\end{aligned}$$

where in the last equality we used the fact that $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$. Thus, we have:

$$\langle l, m|L_\pm L_\mp|l, m\rangle = \hbar^2 l(l+1) - \hbar^2 m^2 \pm \hbar^2 m. \tag{3.7}$$

Putting everything together, we get

$$\langle\hat{L}_x^2\rangle = \frac{\hbar^2}{2}\,[l(l+1) - m^2]. \tag{3.8}$$

By a very similar treatment, we can also compute $\langle\hat{L}_y^2\rangle$ and verify that $\langle\hat{L}_x^2\rangle = \langle\hat{L}_y^2\rangle$.

(c) Using your results, verify the first uncertainty relation above. Can the uncertainty in any two components of $\vec{L}$ ever vanish simultaneously?

Since $\langle\hat{L}_x\rangle = 0$, we have

$$\Delta L_x = \sqrt{\langle\hat{L}_x^2\rangle - \langle\hat{L}_x\rangle^2} = \sqrt{\langle\hat{L}_x^2\rangle} = \frac{\hbar}{\sqrt{2}}\sqrt{l(l+1) - m^2}. \tag{3.9}$$

and similarly for ΔL_y. The left-hand side of the proposed uncertainty relation thus reads

$$\Delta L_x\,\Delta L_y = \frac{\hbar^2}{2}\left[l(l+1) - m^2\right]. \tag{3.10}$$

For a fixed l, this quantity is minimized when m is as large as possible, i.e. when $m = l$. In such a situation we obtain $\Delta L_x \Delta L_y = l\hbar^2/2$. As for the right-hand side, we have

$$\frac{\hbar}{2}\,\langle\hat{L}_z\rangle = \frac{m\hbar^2}{2}. \tag{3.11}$$

This side is maximized precisely when the other side is minimized ($m = l$), giving $\frac{\hbar}{2}\langle \hat{L}_z \rangle = l\hbar^2/2$. In that case the two sides are equal; for all other cases ($m < l$) the left-hand side is greater. We can therefore conclude

$$\Delta L_x \, \Delta L_y \;\geq\; \frac{\hbar}{2} \, \langle \hat{L}_z \rangle. \tag{3.12}$$

It is possible for all components of angular momentum to vanish simultaneously. If a particle is in an eigenstate with $l = m = 0$, then from the relationships proved above we have $\langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle = \langle \hat{L}_z^2 \rangle = 0$ as well as $\langle \hat{L}_x \rangle = \langle \hat{L}_y \rangle = \langle \hat{L}_z \rangle = 0$; consequently $\Delta L_x = \Delta L_y = \Delta L_z = 0$.

4 The Quantum Rigid Rotator

A. Consider a spherically symmetric rigid rotor with moment of inertia $I_x = I_y = I_z = I$. For example, it might help to imagine a person curled up into a compact and uniform sphere and set spinning. Classically, its energy is given by,

$$E = \frac{L^2}{2I}. \tag{4.1}$$

(a) What are the energy eigenstates and eigenvalues for the quantum analog?

The Hamiltonian of the system is simply

$$\hat{H} = \frac{\hat{L}^2}{2I}, \tag{4.2}$$

and clearly commutes with both L^2 and L_z, and thus its eigenstates are the spherical harmonics Y_{lm}, since

$$\hat{L}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}. \tag{4.3}$$

The energy eigenvalues of the quantum rigid rotator are obtained by applying the Hamiltonian on the eigenstates:

$$\hat{H}\, Y_{lm} = \frac{\hat{L}^2}{2I}\, Y_{lm} = \frac{\hbar^2 l(l+1)}{2I}\, Y_{lm} \tag{4.4}$$

and we identify:

$$E_l = \frac{\hbar^2 l(l+1)}{2I}. \tag{4.5}$$

(b) What is the degeneracy of the nth energy eigenvalue? (Degeneracy in quantum mechanics refers to different quantum states having the same energy eigenvalue).

The energy eigenvalues above are labeled only by the total angular momentum quantum number l, and not on the quantum number corresponding to the z-component of the angular momentum. This is the case for all spherically symmetric systems. Now, we recall from the lecture that for a given l the quantum number m runs from $+l$ to $-l$, so every energy eigenvalue is $2l + 1$ degenerate.

B. Let us now imagine that the rotator is stretched a bit such that its moment of inertia in the z-direction becomes $I_z = (1+\epsilon)I$, with the other two moments remaining unchanged.

(a) What are the new energy eigenstates and eigenvalues?

In this case, the Hamiltonian becomes

$$\hat{H} = \frac{\hat{L}_x^2 + \hat{L}_y^2}{2I} + \frac{\hat{L}_z^2}{2I(1+\epsilon)}. \tag{4.6}$$

Using the fact that $L_x^2 + L_y^2 = L^2 - L_z^2$, we find:

$$\hat{H} = \frac{\hat{L}^2 - \hat{L}_z^2}{2I} + \frac{\hat{L}_z^2}{2I(1+\epsilon)} = \frac{\hat{L}^2}{2I} - \frac{\epsilon}{1+\epsilon}\frac{\hat{L}_z^2}{2I}. \tag{4.7}$$

The spherical harmonics Y_{lm} are simultaneous eigenfunctions of both $\hat{L}^2$ and L_z, so the energy eigenfunctions of the above Hamiltonian are again the Y_{lm}. Following a similar procedure as in part (a), we find the energies of this rotator as:

$$E_{lm} = \frac{\hbar^2 l(l+1)}{2I} - \frac{\epsilon}{1+\epsilon}\frac{\hbar^2 m^2}{2I}. \tag{4.8}$$

(b) Sketch the spectrum of energy eigenvalues as a function of ϵ. For what sign of ϵ do the energy eigenvalues get closer together? Intuitively, why?

In Figure 1, we plot the energies in a dimensionless form,

$$\tilde{E}_{l,m}(\epsilon) = l(l+1) - \frac{\epsilon}{1+\epsilon}m^2, \tag{4.9}$$

for $l = 0, 1, 2$.

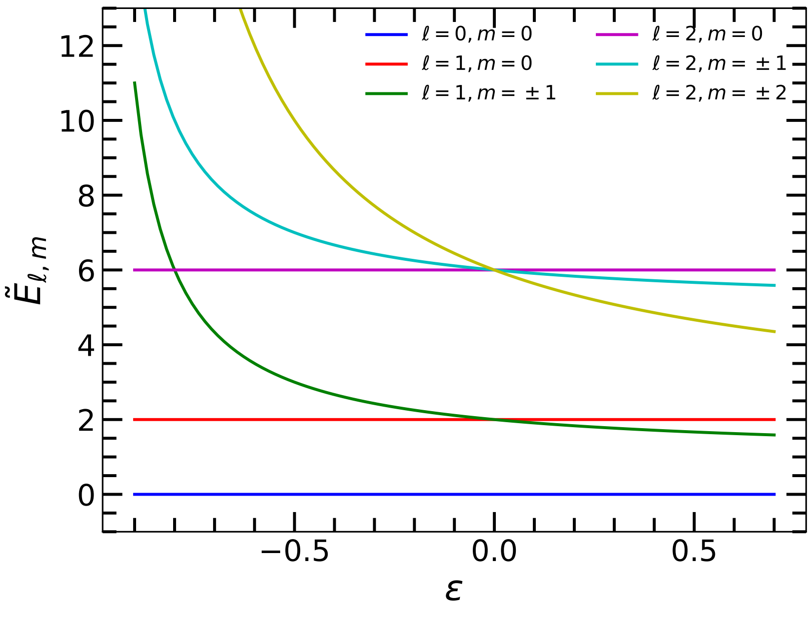


Figure 1: The (dimensionless) energies of the rotator as a function of the parameter ϵ for $l = 0, 1, 2$.

For $m = 0$, there is no dependence on ϵ since the second term in the RHS of the above equation is zero. Let us analyze the spacings between the energy eigenvalues. The difference between two energy eigenstates $E_{l\,m}$ and $E_{l'm'}$ is

$$E_{l\,m} - E_{l'm'} = \frac{\hbar^2\left[l'(l'+1) - l(l+1)\right]}{2I} + \frac{\epsilon}{1+\epsilon}\,\frac{\hbar^2\left(m^2 - m'^2\right)}{2I}. \tag{4.10}$$

The first fraction on the right-hand side is a constant and therefore does not affect the tend of the energy differences; the absolute value of the second fraction shows how close two energies can get, since it depends on ϵ. We always assume that $m^2 > m'^2$ (if not, simply exchange $E_{l\,m} \leftrightarrow E_{l'm'}$), so the important term remaining is $\frac{\epsilon}{1+\epsilon}$. If $\epsilon \geq 0$ then $\left|\frac{\epsilon}{1+\epsilon}\right| \leq \frac{1}{2}$, but if $\epsilon \to -1^+$ then $\left|\frac{\epsilon}{1+\epsilon}\right| \to \infty$; thus, in the former case $\epsilon \geq 0$ the two energies are closer.

Classically this means that for a given L_z the moment of inertia in the z-direction $I_z = (1+\epsilon)I$ increases, the associated rotational energy decreases.

(c) What is the degeneracy of the n^{th} energy eigenvalue? Is the degeneracy partially or fully lifted?

From Eq. (4.8), we see that, for a given l, the $\pm m$ eigenfunctions share the same eigenvalue, so the degeneracy is only partially lifted. This is because the system energy is still invariant under $L_z \to -L_z$, i.e. the rigid rotator starts to spin in the opposite direction but with the same angular-momentum magnitude. The two opposite rotation directions correspond to $+m$ and $-m$.

(d) Now add a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. Does this term lift the degeneracy?

We add an additional term to the Hamiltonian of the form $\hat{H} = -\mathbf{B}\cdot\hat{\boldsymbol{\mu}}$ where $\hat{\boldsymbol{\mu}} = \frac{q}{2m}\hat{\mathbf{L}}$. Thus, the overall Hamiltonian is:

$$\hat{H} = \frac{\hat{L}^2}{2I} - \frac{\epsilon}{1+\epsilon}\frac{\hat{L}_z^2}{2I} + \frac{q}{2m}\mathbf{B}\cdot\hat{\mathbf{L}}. \tag{4.11}$$

The magnetic field is oriented along the z-axis and thus the Hamiltonian reduces to:

$$\hat{H} = \frac{\hat{L}^2}{2I} - \frac{\epsilon}{1+\epsilon}\frac{\hat{L}_z^2}{2I} + \frac{q}{2m}B_z\hat{L}_z. \tag{4.12}$$

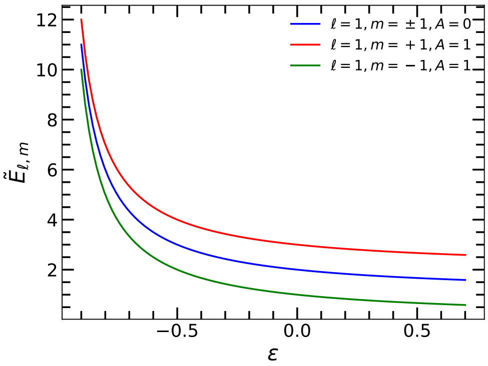
This Hamiltonian commutes with both L^2 and L_z, and thus its eigenstates are the usual $|l,m\rangle$ (or the spherical harmonics in position representation. The energies are thus readily obtained:

$$E_{l,m}(\epsilon) = \frac{\hbar^2 l(l+1)}{2I} - \frac{\epsilon}{1+\epsilon}\frac{\hbar^2 m^2}{2I} + \frac{q}{2m}B_z\hbar m, \tag{4.13}$$

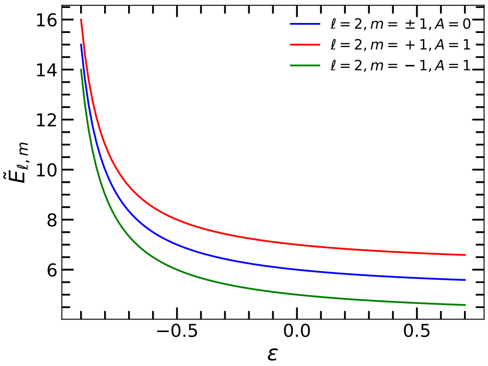
which can be re-written in the following dimensionless form:

$$\tilde{E}_{l,m}(\epsilon) = l(l+1) - \frac{\epsilon}{1+\epsilon}m^2 + A\,m, \quad A = \frac{qB_z 2I}{2m\hbar} \tag{4.14}$$

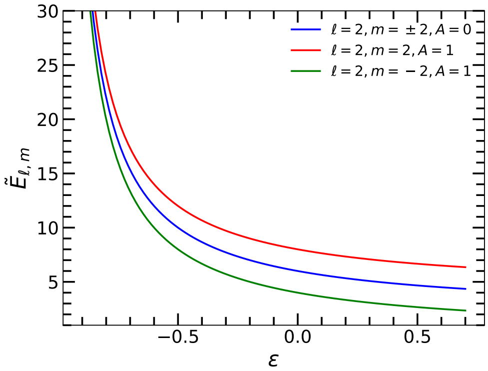
where A is a dimensionless constant and is directly proportional to the magnitude of the magnetic field. This term clearly breaks the symmetry $L_z \to -L_z$, and thus degeneracy is lifted. In Figure 2, we demonstrate the breaking of the degeneracy for $\pm m$ when $B \neq 0$.



(a) $l = 1, m = \pm 1$



(b) $l = 2, m = \pm 1$



(c) $l = 2, m = \pm 2$

Figure 2: The energies of the rotator as a function of the parameter ϵ for a non-zero magnetic field.